On the computation of preliminary orbits for space debris with radar observations

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Abstract

We introduce a new method to perform preliminary orbit determination for space debris on low Earth orbits (LEO). This method works with tracks of radar observations: each track is composed by $n \geq 4$ topocentric position vectors per pass of the satellite, taken at very short time intervals. We assume very accurate values for the range ρ, while the angular positions (i.e. the line of sight, given by the pointing of the antenna) are less accurate. We wish to correct the errors in the angular positions already in the computation of a preliminary orbit. With the information contained in a pair of radar tracks, using the laws of the two-body dynamics, we can write 8 equations in 8 unknowns. The unknowns are the components of the topocentric velocity orthogonal to the line of sight at the two mean epochs of the tracks, and the corrections $\boldsymbol{\Delta}$ to be applied to the angular positions. We take advantage of the fact that the components of $\boldsymbol{\Delta}$ are typically small. We show the results of some tests, performed with simulated observations, and compare this algorithm with Gibbs' method and the Keplerian integrals method.

1 Introduction

We investigate the preliminary orbit determination problem for a satellite of the Earth using radar observations collected by an instrument with given technical

specifications, and with a fixed observation scheduling. Assume we collect the following data for the observed object:

$$(t_j, \rho_j, \alpha_j, \delta_j), \qquad j = 1 \dots 4 \tag{1}$$

where the triples $(\rho_j, \alpha_j, \delta_j)$ represent topocentric spherical coordinates of the object at epochs t_j. Typically α_j, δ_j are the values of right ascension and declination. We shall call *radar track* the set of observations in (1).

The following assumptions will be made on the data composing the tracks. The time difference $t_{j+1} - t_j$ between consecutive observations is $\Delta t = 10$ s. The range data ρ_j are very precise: the statistical error in the range is given by its RMS σ_ρ, which is 10 m. On the other hand we assume that the angles α_j, δ_j are not precisely determined: their RMS σ_α, σ_δ are supposed to be 0.2 degrees.

Given a radar track we can compute by interpolation the following data:

$$(\bar{t}, \bar{\alpha}, \bar{\delta}, \rho, \dot{\rho}, \ddot{\rho}). \tag{2}$$

Here $\bar{t}$, $\bar{\alpha}$ and $\bar{\delta}$ are the mean values of the epoch and the angles, and ρ, $\dot{\rho}$, $\ddot{\rho}$ are the values of a function $\rho(t)$ and its derivatives at $t = \bar{t}$, where $\rho(t)$ is given by a quadratic fit with the (t_j, ρ_j) data.

For low Earth orbits (LEO) these assumptions imply that the interpolated values of $\dot{\alpha}$, $\dot{\delta}$ are very badly accurate, to the point that their value can be of the same order of the errors, therefore they are practically undetermined.

By the above considerations, given a vector (2) obtained by a radar track, and using spherical coordinates and velocities

$$(\rho, \alpha, \delta, \dot{\rho}, \dot{\alpha}, \dot{\delta})$$

to describe the orbit, we can consider as unknowns the quantities $(\Delta\alpha, \Delta\delta, \dot{\alpha}, \dot{\delta})$, with

$$\alpha = \bar{\alpha} + \Delta\alpha, \qquad \delta = \bar{\delta} + \Delta\delta,$$

where $\Delta\alpha, \Delta\delta$ are small deviations from the mean values $\bar{\alpha}$, $\bar{\delta}$.

To search for the values of the unknowns we need to use additional data: we can try to use the data of 2 radar tracks, together with a dynamical model, to compute one or more preliminary orbits. This is a *linkage problem*, see [5].

In this paper we propose a new algorithm for the linkage, which takes advantage of the smallness of $\Delta\alpha, \Delta\delta$, that we call *infinitesimal angles*. We write the equations for preliminary orbits by using the 5 algebraic integrals of Kepler's

problem, Lambert's equation for elliptic motion (see Section 12) and the projection of the equations of motion along the line of sight.
Moreover, we perform some tests to compare this method with Gibbs' method, using only one radar track, and with the Keplerian integrals method, which solves a linkage problem using $(\bar{\alpha}, \bar{\delta}, \rho, \dot{\rho})$ at two mean epochs (see [9], [2], [3]).

The paper is organized as follows. First we introduce some notation and recall the basic results on Kepler's motion which are relevant for this work (see Sections 2, 3, 4). The equations for the linkage problem, see (7), are presented in Section 5, and in Sections 6, 7 we show two different ways to compute solutions of (7). In Section 9 we present the results of some numerical tests, including a comparison with the already known methods recalled in Section 8. Finally, in Section 12, we recall the proof of Lambert's theorem for elliptic orbits and give a geometrical interpretation of the results. Moreover, we show a method to correct the observations of a radar track so that they correspond to points in the same plane.

2 The equations of motion

Let us denote by $\mathbf{e}^\rho$ the unit vector corresponding to the line of sight, and by $\mathbf{q}$ the geocentric position of the observer. Then the position of the observed body is $\mathbf{r} = \mathbf{q} + \rho\mathbf{e}^\rho$, where ρ is the range. Using the right ascension α and the declination δ as coordinates we have

$$\mathbf{e}^\rho = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta).$$

We assume the observed body is moving according to Newton's equations

$$\ddot{\mathbf{r}} = -\frac{\mu}{|\mathbf{r}|^3}\mathbf{r}. \tag{3}$$

We introduce the moving frame $\{\mathbf{e}^\rho, \hat{\mathbf{v}}, \hat{\mathbf{n}}\}$, depending on the epoch t, where $\hat{\mathbf{v}} = \frac{d}{ds}\mathbf{e}^\rho$, regarding $\mathbf{e}^\rho$ as function of the arc-length s, and $\hat{\mathbf{n}} = \mathbf{e}^\rho \times \hat{\mathbf{v}}$. By projecting equation (3) on these vectors we obtain

$$\begin{cases} \ddot{\rho} - \rho\eta^2 + \ddot{\mathbf{q}}\cdot\mathbf{e}^\rho & = -\dfrac{\mu}{|\mathbf{r}|^3}(\mathbf{r}\cdot\mathbf{e}^\rho) \\ 2\dot{\rho}\eta + \rho\dot{\eta} + \ddot{\mathbf{q}}\cdot\hat{\mathbf{v}} & = -\dfrac{\mu}{|\mathbf{r}|^3}(\mathbf{r}\cdot\hat{\mathbf{v}}) \\ \kappa\eta^2\rho + \ddot{\mathbf{q}}\cdot\hat{\mathbf{n}} & = -\dfrac{\mu}{|\mathbf{r}|^3}(\mathbf{r}\cdot\hat{\mathbf{n}}) \end{cases}$$

where $\eta = \sqrt{\dot\alpha^2\cos^2\delta + \dot\delta^2}$ is the proper motion and $\kappa = \frac{d}{ds}\hat{\mathbf{v}}\cdot\hat{\mathbf{n}}$. For later use we introduce the notation

$$\mathcal{K} = \Big(\ddot{\mathbf{r}} + \frac{\mu}{|\mathbf{r}|^3}\mathbf{r}\Big)\cdot\mathbf{e}^\rho = \ddot\rho - \rho\eta^2 + \ddot{\mathbf{q}}\cdot\mathbf{e}^\rho + \frac{\mu}{|\mathbf{r}|^3}(\mathbf{r}\cdot\mathbf{e}^\rho).$$

3 The two-body integrals

We write below (see also [3]) the expressions of the first integrals of Kepler's problem, i.e. the angular momentum $\mathbf{c}$, the energy $\mathcal{E}$ and the Laplace-Lenz vector $\mathbf{L}$, in the variables $\rho, \alpha, \delta, \dot\rho, \xi, \zeta$, with

$$\xi = \rho\dot\alpha\cos\delta, \qquad \zeta = \rho\dot\delta. \tag{4}$$

We have

$$\begin{aligned}
&\mathbf{c} = \mathbf{A}\xi + \mathbf{B}\zeta + \mathbf{C},\\
&\mathcal{E} = \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|},\\
&\mu\mathbf{L}(\rho,\dot\rho) = \dot{\mathbf{r}}\times\mathbf{c} - \mu\frac{\mathbf{r}}{|\mathbf{r}|} = \Big(|\dot{\mathbf{r}}|^2 - \frac{\mu}{|\mathbf{r}|}\Big)\mathbf{r} - (\dot{\mathbf{r}}\cdot\mathbf{r})\dot{\mathbf{r}},
\end{aligned}$$

where

$$\mathbf{A} = \mathbf{r}\times\mathbf{e}^\alpha, \quad \mathbf{B} = \mathbf{r}\times\mathbf{e}^\delta, \quad \mathbf{C} = \mathbf{r}\times\dot{\mathbf{q}} + \dot\rho\,\mathbf{q}\times\mathbf{e}^\rho,$$

with

$$\mathbf{e}^\alpha = \frac{1}{\cos\delta}\frac{\partial\mathbf{e}^\rho}{\partial\alpha}, \qquad \mathbf{e}^\delta = \frac{\partial\mathbf{e}^\rho}{\partial\delta},$$

and

$$\begin{aligned}
&\dot{\mathbf{r}} = \xi\mathbf{e}^\alpha + \zeta\mathbf{e}^\delta + (\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}}),\\
&|\dot{\mathbf{r}}|^2 = \xi^2 + \zeta^2 + 2\dot{\mathbf{q}}\cdot\mathbf{e}^\alpha\xi + 2\dot{\mathbf{q}}\cdot\mathbf{e}^\delta\zeta + |\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}}|^2,\\
&\dot{\mathbf{r}}\cdot\mathbf{r} = \mathbf{q}\cdot\mathbf{e}^\alpha\xi + \mathbf{q}\cdot\mathbf{e}^\delta\zeta + (\dot\rho\mathbf{e}^\rho + \dot{\mathbf{q}})\cdot\mathbf{r}.
\end{aligned}$$

We introduce the notation

$$q^\alpha = \mathbf{q}\cdot\mathbf{e}^\alpha, \quad q^\delta = \mathbf{q}\cdot\mathbf{e}^\delta, \quad \dot q^\alpha = \dot{\mathbf{q}}\cdot\mathbf{e}^\alpha, \quad \dot q^\delta = \dot{\mathbf{q}}\cdot\mathbf{e}^\delta.$$

Note that $\xi^2 + \zeta^2 = \rho^2\eta^2$.

4 Lambert's equation

Lambert's theorem for elliptic motion gives the following relation for the orbital elements of a body on a Keplerian orbit at epochs t_1, t_2:

$$n(t_2 - t_1) = \beta - \gamma - (\sin\beta - \sin\gamma) + 2k\pi. \tag{5}$$

Here $k \in \mathbb{N}$ is the number of revolutions in the time interval $[t_1, t_2]$, $n = n(a)$ is the mean motion, where $a = -\mu/(2\mathcal{E})$ (the energy is the same at the two epochs), and the angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r_1 + r_2 + d}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r_1 + r_2 - d}{4a}, \tag{6}$$

and

$$0 \leq \beta - \gamma \leq 2\pi,$$

with r_1, r_2 the distances from the center of force, and d the length of the chord joining the two positions of the body at epochs t_1, t_2. For a fixed number of revolutions we have 4 different choices for the pairs (β, γ), see Section 12 and [1] for the details.

5 Linkage

We wish to link two sets of radar data (2), with mean epochs $\bar{t}_i$, $i = 1, 2$, and compute one or more preliminary orbits. In the following we use labels 1, 2 for the quantities introduced in Sections 2, 3, 4 according to the epoch.

Let us denote by $\mathcal{L}$ the expression defining Lambert's equation. More precisely, $\mathcal{L} = 0$ is one of the possible cases occurring in (5), see Section 12. Moreover, let us define $\mathbf{v}_2 = \mathbf{e}_2^\rho \times \mathbf{q}_2$. We consider the system

$$(\mathbf{c}_1 - \mathbf{c}_2, \mathcal{E}_1 - \mathcal{E}_2, \mathcal{K}_1, \mathcal{K}_2, (\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v}_2, \mathcal{L}) = \mathbf{0} \tag{7}$$

of 8 equations in the 8 unknowns $(\mathbf{X}, \mathbf{\Delta})$, with

$$\mathbf{X} = (\xi_1, \zeta_1, \xi_2, \zeta_2), \qquad \mathbf{\Delta} = (\Delta\alpha_1, \Delta\delta_1, \Delta\alpha_2, \Delta\delta_2).$$

Note that the unknowns are divided into 2 sets so that $\mathbf{\Delta}$ is the vector of infinitesimal angles. To solve system (7) we first compute $\mathbf{X}$ as function of $\mathbf{\Delta}$ using 4 of these equations, then we substitute $\mathbf{X}(\mathbf{\Delta})$ into the remaining equations and search for solutions of the resulting nonlinear system by applying Newton-Raphson's method. Taking advantage of the assumed smallness of the solutions $\mathbf{\Delta}$, we can use $\mathbf{\Delta} = \mathbf{0}$ as starting guess.

6 Computing $\mathbf{X}(\boldsymbol{\Delta})$

We describe below two methods to compute $\mathbf{X}$ as function of $\boldsymbol{\Delta}$ using some of the equations of system (7). One approach uses linear equations, see Section 6.1, while the equations for the other are quadratic, see Section 6.2.

6.1 Linear equations

Substituting $2\mathcal{E}_1 + \rho_1\mathcal{K}_1 - 2\mathcal{E}_2 - \rho_2\mathcal{K}_2$ in place of $\mathcal{E}_1 - \mathcal{E}_2$ in (7) we obtain an equivalent system and the equation

$$2\mathcal{E}_1 + \rho_1\mathcal{K}_1 = 2\mathcal{E}_2 + \rho_2\mathcal{K}_2 \tag{8}$$

is linear in the variables $\mathbf{X} = (\xi_1, \zeta_1, \xi_2, \zeta_2)$.
Using equation (8) and the conservation of the angular momentum we obtain a linear system in the variables $\mathbf{X}$:

$$\mathcal{M}\mathbf{X} = \mathbf{V}. \tag{9}$$

Here

$$\mathcal{M} = \begin{bmatrix} A_{11} & B_{11} & -A_{21} & -B_{21} \\ A_{12} & B_{12} & -A_{22} & -B_{22} \\ A_{13} & B_{13} & -A_{23} & -B_{23} \\ \dot{q}_1^\alpha & \dot{q}_1^\delta & -\dot{q}_2^\alpha & -\dot{q}_2^\delta \end{bmatrix},$$

where A_{ij}, B_{ij} are the components of $\mathbf{A}_i, \mathbf{B}_i$, and $\dot{q}_i^\alpha = \dot{\mathbf{q}}_i \cdot \mathbf{e}_i^\alpha$, $\dot{q}_i^\delta = \dot{\mathbf{q}}_i \cdot \mathbf{e}_i^\delta$, for $i = 1, 2$.
Moreover

$$\mathbf{V} = (C_{21} - C_{11}, C_{22} - C_{12}, C_{23} - C_{13}, D_2 - D_1)^T,$$

where C_{ij} are the components of $\mathbf{C}_i$ and

$$D_i = \frac{1}{2}\Big(\rho_i^2\eta_i^2 + |\dot{\rho}_i\mathbf{e}_i^\rho + \dot{\mathbf{q}}_i|^2\Big) - \frac{\mu}{|\mathbf{r}_i|},$$

with η_i^2 expressed as function of $(\Delta\alpha_i, \Delta\delta_i)$ by using the equations $\mathcal{K}_i = 0$, $i = 1, 2$, that is using relation

$$\eta^2 = \frac{1}{\rho}\Big(\ddot{\rho} + \ddot{\mathbf{q}} \cdot \mathbf{e}^\rho + \frac{\mu}{|\mathbf{r}|^3}(\mathbf{r} \cdot \mathbf{e}^\rho)\Big)$$

at the 2 epochs $\bar{t}_1, \bar{t}_2$.

We can write $\mathbf{X}$ as function of $\boldsymbol{\Delta}$ by solving system (9).
Let us call $\mathcal{M}_{hj}$ the components of $\mathcal{M}$, and V_h the components of $\mathbf{V}$. The solutions of (9) are given by

$$\xi_i = \frac{|\mathcal{M}_{2i-1}|}{|\mathcal{M}|}, \qquad \zeta_i = \frac{|\mathcal{M}_{2i}|}{|\mathcal{M}|}, \qquad i = 1, 2 \tag{10}$$

where $\mathcal{M}_k$ has components

$$\mathcal{M}^{(k)}_{hj} = \begin{cases} \mathcal{M}_{hj} & \text{if } k \neq j \\ V_h & \text{if } k = j \end{cases}$$

and $|\mathcal{M}|$, $|\mathcal{M}_k|$ represent the determinants of $\mathcal{M}$, $\mathcal{M}_k$.

6.2 Quadratic equations

The orbits at epochs $\bar{t}_1$, $\bar{t}_2$, computed with the solution $\mathbf{X}$ of system (9), do not necessarily share the same energy $\mathcal{E}$. This can produce some problems in the linear algorithm described above, especially when solving Lambert's equation, where the right-hand sides of (6) may become greater that 1 during the iterations of Newton-Raphson's method. We can force the orbits to share the same energy by solving the first 4 equations in (7), that are quadratic equations in the variable $\mathbf{X}$.
By introducing the vector

$$\mathbf{Y} = (\xi_1, \zeta_1, \xi_2),$$

we can write the conservation of the angular momentum as the linear system

$$\mathcal{N}\mathbf{Y} = \mathbf{W}. \tag{11}$$

Here

$$\mathcal{N} = \begin{bmatrix} A_{11} & B_{11} & -A_{21} \\ A_{12} & B_{12} & -A_{22} \\ A_{13} & B_{13} & -A_{23} \end{bmatrix} \tag{12}$$

and

$$\mathbf{W} = \zeta_2 \mathbf{W}^{(1)} + \mathbf{W}^{(0)},$$

where

$$\begin{aligned} \mathbf{W}^{(1)} &= (B_{21}, B_{22}, B_{23})^T, \\ \mathbf{W}^{(0)} &= (C_{21} - C_{11}, C_{22} - C_{12}, C_{23} - C_{13})^T. \end{aligned}$$

We solve system (11). Let us call $\mathcal{N}_{hj}$ the components of $\mathcal{N}$ and W_h,$W_h^{(0)}$,$W_h^{(1)}$ the components of $\mathbf{W}$,$\mathbf{W}^{(0)}$,$\mathbf{W}^{(1)}$. The solutions of (11) are functions of ζ_2, $\mathbf{\Delta}$, and are given by

$$\tilde{\xi}_1 = \frac{|\mathcal{N}_1|}{|\mathcal{N}|}, \quad \tilde{\zeta}_1 = \frac{|\mathcal{N}_2|}{|\mathcal{N}|}, \quad \tilde{\xi}_2 = \frac{|\mathcal{N}_3|}{|\mathcal{N}|},$$

where $\mathcal{N}_k$ has components

$$\mathcal{N}_{hj}^{(k)} = \begin{cases} \mathcal{N}_{hj} & \text{if } k \neq j \\ W_h & \text{if } k = j \end{cases}.$$

From the conservation of energy we can find ζ_2 as function of $\mathbf{\Delta}$. We write

$$F_2\zeta_2^2 + F_1\zeta_2 + F_0 = 0, \tag{13}$$

with

$$\begin{aligned} F_2 &= \frac{1}{|\mathcal{N}|^2}(|\mathcal{N}_1^{(1)}|^2 + |\mathcal{N}_2^{(1)}|^2 - |\mathcal{N}_3^{(1)}|^2) - 1 \\ F_1 &= \frac{2}{|\mathcal{N}|^2}(|\mathcal{N}_1^{(1)}||\mathcal{N}_1^{(0)}| + |\mathcal{N}_2^{(1)}||\mathcal{N}_2^{(0)}| - |\mathcal{N}_3^{(1)}||\mathcal{N}_3^{(0)}|) + \\ &+ \frac{2}{|\mathcal{N}|}(\dot{q}_1^\alpha|\mathcal{N}_1^{(1)}| + \dot{q}_1^\delta|\mathcal{N}_2^{(1)}| - \dot{q}_2^\alpha|\mathcal{N}_3^{(1)}| - \dot{q}_2^\delta|\mathcal{N}|) \\ F_0 &= \frac{1}{|\mathcal{N}|^2}(|\mathcal{N}_1^{(0)}|^2 + |\mathcal{N}_2^{(0)}|^2 - |\mathcal{N}_3^{(0)}|^2) \\ &+ \frac{2}{|\mathcal{N}|}(\dot{q}_1^\alpha|\mathcal{N}_1^{(0)}| + \dot{q}_1^\delta|\mathcal{N}_2^{(0)}| - \dot{q}_2^\alpha|\mathcal{N}_3^{(0)}|) \\ &+ \mathfrak{D}_1 - \mathfrak{D}_2, \end{aligned}$$

where $\mathcal{N}_k^{(\ell)}$, $k = 1, 2, 3$, $\ell = 0, 1$, has components

$$\mathcal{N}_{hj}^{(k,\ell)} = \begin{cases} \mathcal{N}_{hj} & \text{if } k \neq j \\ W_h^{(\ell)} & \text{if } k = j \end{cases},$$

and

$$\mathfrak{D}_i = 2D_i - \rho_i^2\eta_i^2, \qquad i = 1, 2.$$

Therefore we have

$$\begin{aligned} \xi_1(\mathbf{\Delta}) &= \tilde{\xi}_1(\zeta_2(\mathbf{\Delta}), \mathbf{\Delta}), \\ \zeta_1(\mathbf{\Delta}) &= \tilde{\zeta}_1(\zeta_2(\mathbf{\Delta}), \mathbf{\Delta}), \\ \xi_2(\mathbf{\Delta}) &= \tilde{\xi}_2(\zeta_2(\mathbf{\Delta}), \mathbf{\Delta}), \end{aligned}$$

where $\zeta_2(\mathbf{\Delta})$ is a solution of (13). Note that we can have up to two acceptable expressions for $\mathbf{X}(\mathbf{\Delta})$.

7 Computing Δ

We introduce the vector

$$\mathbf{G} = (\mathcal{K}_1, \mathcal{K}_2, (\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v}_2, \mathcal{L}).$$

Remark 1 *To select the relevant expressions of $\mathcal{L}$ we need to guess the value of k in (5). We can do this by assuming $\boldsymbol{\Delta} = \mathbf{0}$ and computing the possible orbits according to the linear or quadratic equations for $\mathbf{X}(\boldsymbol{\Delta})$. In both cases we obtain two possible values for the number of revolutions k: with the linear equations we can obtain two different values of k at the two epochs $\bar{t}_1$, $\bar{t}_2$; with the quadratic equations we may obtain two orbits with different k at the same epoch, say $\bar{t}_1$, but from conservation of energy we obtain the same values at $\bar{t}_2$.*

By substituting the possible expressions of $\mathbf{X}(\boldsymbol{\Delta})$, coming from either the linear or the quadratic equations, we obtain the reduced system

$$\mathcal{G}(\boldsymbol{\Delta}) = \mathbf{G}(\mathbf{X}(\boldsymbol{\Delta}), \boldsymbol{\Delta}) = \mathbf{0}. \tag{14}$$

Since the unknowns in $\boldsymbol{\Delta}$ are small, we can try to apply Newton-Raphson's method with $\boldsymbol{\Delta} = \mathbf{0}$ as starting guess. Thus we try to compute an approximation for $\boldsymbol{\Delta}$ by the iterative formula

$$\boldsymbol{\Delta}_{k+1} = \boldsymbol{\Delta}_k - \Big[\frac{\partial \mathcal{G}}{\partial \boldsymbol{\Delta}}(\boldsymbol{\Delta}_k)\Big]^{-1} \mathcal{G}(\boldsymbol{\Delta}_k), \qquad \boldsymbol{\Delta}_0 = \mathbf{0}. \tag{15}$$

Equations (15) are linear, and are defined by (14) and by the Jacobian matrix

$$\frac{\partial \mathcal{G}}{\partial \boldsymbol{\Delta}}(\boldsymbol{\Delta}_k) = \frac{\partial \mathbf{G}}{\partial \mathbf{X}}(\mathbf{X}_k, \boldsymbol{\Delta}_k)\frac{\partial \mathbf{X}}{\partial \boldsymbol{\Delta}}(\boldsymbol{\Delta}_k) + \frac{\partial \mathbf{G}}{\partial \boldsymbol{\Delta}}(\mathbf{X}_k, \boldsymbol{\Delta}_k),$$

with $\mathbf{X}_k = \mathbf{X}(\boldsymbol{\Delta}_k)$.

Remark 2 *Note that at each iteration the number of solutions can be doubled, but if we impose the value of $\boldsymbol{\Delta}_{k+1}$ to be close to $\boldsymbol{\Delta}_k$ then we can usually avoid bifurcations.*

The computation of the Jacobian matrix $\frac{\partial \mathcal{G}}{\partial \boldsymbol{\Delta}}$ is described below, enhancing the differences between the linear and the quadratic case.

7.1 The derivatives $\frac{\partial \mathbf{G}}{\partial \mathbf{X}}$

$$
\begin{aligned}
\frac{\partial \mathcal{K}_1}{\partial \mathbf{X}} &= -\frac{2}{\rho_1}(\xi_1, \zeta_1, 0, 0) \\
\frac{\partial \mathcal{K}_2}{\partial \mathbf{X}} &= -\frac{2}{\rho_2}(0, 0, \xi_2, \zeta_2)
\end{aligned}
$$

We observe that

$$\mathbf{L}_2 \cdot \mathbf{v}_2 = -\frac{1}{\mu}(\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)(\dot{\mathbf{r}}_2 \cdot \mathbf{v}_2).$$

Thus we have

$$
\begin{aligned}
\frac{\partial}{\partial \xi_1}[(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v}_2] &= \frac{2}{\mu}(\xi_1 + \dot{\mathbf{q}}_1 \cdot \mathbf{e}_1^{\alpha})(\mathbf{r}_1 \cdot \mathbf{v}_2) \\
&\quad -\frac{1}{\mu}[(\mathbf{q}_1 \cdot \mathbf{e}_1^{\alpha})(\dot{\mathbf{r}}_1 \cdot \mathbf{v}_2) + (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)(\mathbf{e}_1^{\alpha} \cdot \mathbf{v}_2)] \\
\frac{\partial}{\partial \zeta_1}[(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v}_2] &= \frac{2}{\mu}(\zeta_1 + \dot{\mathbf{q}}_1 \cdot \mathbf{e}_1^{\delta})(\mathbf{r}_1 \cdot \mathbf{v}_2) \\
&\quad -\frac{1}{\mu}[(\mathbf{q}_1 \cdot \mathbf{e}_1^{\delta})(\dot{\mathbf{r}}_1 \cdot \mathbf{v}_2) + (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)(\mathbf{e}_1^{\delta} \cdot \mathbf{v}_2)] \\
\frac{\partial}{\partial \xi_2}[(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v}_2] &= \frac{1}{\mu}[(\mathbf{q}_2 \cdot \mathbf{e}_2^{\alpha})(\dot{\mathbf{r}}_2 \cdot \mathbf{v}_2) + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)(\mathbf{e}_2^{\alpha} \cdot \mathbf{v}_2)] \\
\frac{\partial}{\partial \zeta_2}[(\mathbf{L}_1 - \mathbf{L}_2) \cdot \mathbf{v}_2] &= \frac{1}{\mu}[(\mathbf{q}_2 \cdot \mathbf{e}_2^{\delta})(\dot{\mathbf{r}}_2 \cdot \mathbf{v}_2) + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)(\mathbf{e}_2^{\delta} \cdot \mathbf{v}_2)]
\end{aligned}
$$

For Lambert's equation, the derivatives are given by

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{X}} &= \frac{\partial n}{\partial \mathbf{X}}(\bar{t}_1 - \bar{t}_2) + \frac{\partial(\beta - \sin\beta)}{\partial \mathbf{X}} - \frac{\partial(\gamma - \sin\gamma)}{\partial \mathbf{X}}, \\
\frac{\partial n}{\partial \mathbf{X}} &= -\frac{3}{2\mu}\sqrt{-2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \mathbf{X}}, \\
\frac{\partial(\beta - \sin\beta)}{\partial \mathbf{X}} &= (1 - \cos\beta)\frac{\partial \beta}{\partial \mathbf{X}} = 2\sqrt{\frac{\Gamma_+}{1 - \Gamma_+}}\frac{\partial \Gamma_+}{\partial \mathbf{X}}, \\
\frac{\partial(\gamma - \sin\gamma)}{\partial \mathbf{X}} &= (1 - \cos\gamma)\frac{\partial \gamma}{\partial \mathbf{X}} = 2\sqrt{\frac{\Gamma_-}{1 - \Gamma_-}}\frac{\partial \Gamma_-}{\partial \mathbf{X}},
\end{aligned}
$$

with

$$\begin{aligned}
\Gamma_+ &= \sin^2\frac{\beta}{2} = -\frac{r_1+r_2+d}{2\mu}\mathcal{E}_1,\\
\Gamma_- &= \sin^2\frac{\gamma}{2} = -\frac{r_1+r_2-d}{2\mu}\mathcal{E}_1.
\end{aligned}$$

In the expression for $\frac{\partial n}{\partial \mathbf{X}}$ we use the energy $\mathcal{E}_1$ at epoch $\bar{t}_1$. We could as well choose $\mathcal{E}_2$ at epoch $\bar{t}_2$: this choice is arbitrary in the linear case, in fact computing $\mathbf{X}(\mathbf{\Delta})$ with the linear algorithm, we generally have $\mathcal{E}_1(\xi_1(\mathbf{\Delta}),\zeta_1(\mathbf{\Delta})) \neq \mathcal{E}_2(\xi_2(\mathbf{\Delta}),\zeta_2(\mathbf{\Delta}))$.

Since r_1, r_2, d do not depend on $\mathbf{X}$, we have

$$\begin{aligned}
\frac{\partial\Gamma_+}{\partial\mathbf{X}} &= -\frac{r_1+r_2+d}{2\mu}\frac{\partial\mathcal{E}_1}{\partial\mathbf{X}},\\
\frac{\partial\Gamma_-}{\partial\mathbf{X}} &= -\frac{r_1+r_2-d}{2\mu}\frac{\partial\mathcal{E}_1}{\partial\mathbf{X}},
\end{aligned}$$

with

$$\frac{\partial\mathcal{E}_1}{\partial\mathbf{X}} = ((\xi_1+\mathbf{q}_1\cdot\mathbf{e}_1^\alpha),(\zeta_1+\mathbf{q}_1\cdot\mathbf{e}_1^\delta),0,0).$$

7.2 The derivatives $\frac{\partial\mathbf{X}}{\partial\mathbf{\Delta}}$

To compute derivatives with respect to $\mathbf{\Delta}$ we use as intermediate variables the unit vectors $\mathbf{e}_j^\rho, \mathbf{e}_j^\alpha, \mathbf{e}_j^\delta$, $j=1,2$. To this aim we introduce the vector

$$\mathbf{E} = \begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\end{bmatrix}, \qquad \text{where} \quad \mathbf{E}_j = \begin{pmatrix}\mathbf{e}_j^\rho\\ \mathbf{e}_j^\alpha\\ \mathbf{e}_j^\delta\end{pmatrix}.$$

Its derivatives with respect to $\mathbf{\Delta}$ are given by

$$\frac{\partial\mathbf{E}}{\partial\mathbf{\Delta}} = \begin{bmatrix}\frac{\partial\mathbf{E}_1}{\partial(\alpha_1,\delta_1)} & \mathbf{0}\\ \mathbf{0} & \frac{\partial\mathbf{E}_2}{\partial(\alpha_2,\delta_2)}\end{bmatrix},$$

where

$$\frac{\partial\mathbf{E}_j}{\partial(\alpha_j,\delta_j)} = \begin{bmatrix}\cos\delta_j\mathbf{e}_j^\alpha & \mathbf{e}_j^\delta\\ \mathbf{e}_j^\perp & \mathbf{0}\\ -\sin\delta_j\mathbf{e}_j^\alpha & -\mathbf{e}_j^\rho\end{bmatrix}$$

and $\mathbf{e}_j^\perp = -(\cos\alpha_j, \sin\alpha_j, 0)^T$.

Moreover, we need to compute $\frac{\partial\mathbf{X}}{\partial\mathbf{E}}$. We describe the different procedures for the linear and quadratic methods.

7.2.1 The derivatives $\frac{\partial \mathbf{X}}{\partial \mathbf{E}}$, linear case

Using (10), we only need to compute

$$\begin{aligned}\frac{\partial \xi_h}{\partial \mathbf{E}} &= \frac{1}{|\mathcal{M}|}\frac{\partial |\mathcal{M}_{2h-1}|}{\partial \mathbf{E}} - \frac{|\mathcal{M}_{2h-1}|}{|\mathcal{M}|^2}\frac{\partial |\mathcal{M}|}{\partial \mathbf{E}},\\ \frac{\partial \zeta_h}{\partial \mathbf{E}} &= \frac{1}{|\mathcal{M}|}\frac{\partial |\mathcal{M}_{2h}|}{\partial \mathbf{E}} - \frac{|\mathcal{M}_{2h}|}{|\mathcal{M}|^2}\frac{\partial |\mathcal{M}|}{\partial \mathbf{E}}.\end{aligned}$$

We take advantage of the following relation, valid for any matrix A of order n with coefficients a_{ij} depending on a variable x:

$$\frac{d}{dx}|A| = \sum_{h=1}^{n} |B_h|,$$

where B_h has coefficients $b_{ij}^{(h)}$, with

$$b_{ij}^{(h)} = \begin{cases} a_{ij} & h \neq j \\ \dfrac{d}{dx}a_{ij} & h = j \end{cases}.$$

7.2.2 The derivatives $\frac{\partial \mathbf{X}}{\partial \mathbf{E}}$, quadratic case

From the implicit function theorem applied to equation (13) we obtain

$$\frac{\partial \zeta_2}{\partial \mathbf{E}} = -\Big[\frac{1}{(2F_2\zeta_2 + F_1)}\Big(\frac{\partial F_2}{\partial \mathbf{E}}\zeta_2^2 + \frac{\partial F_1}{\partial \mathbf{E}}\zeta_2 + \frac{\partial F_0}{\partial \mathbf{E}}\Big)\Big]\Big|_{\zeta_2=\zeta_2^{(i)}(\mathbf{E})}.$$

Let us define

$$\begin{aligned}\xi_1(\mathbf{E}) &= \tilde{\xi}_1(\zeta_2(\mathbf{E}), \mathbf{E}),\\ \zeta_1(\mathbf{E}) &= \tilde{\zeta}_1(\zeta_2(\mathbf{E}), \mathbf{E}),\\ \xi_2(\mathbf{E}) &= \tilde{\xi}_2(\zeta_2(\mathbf{E}), \mathbf{E}).\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial \xi_1}{\partial \mathbf{E}} &= \frac{1}{|\mathcal{N}|}\Big(\frac{\partial |\mathcal{N}_1|}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial \mathbf{E}} + \frac{\partial |\mathcal{N}_1|}{\partial \mathbf{E}}\Big) - \frac{|\mathcal{N}_1|}{|\mathcal{N}|^2}\frac{\partial |\mathcal{N}|}{\partial \mathbf{E}},\\ \frac{\partial \zeta_1}{\partial \mathbf{E}} &= \frac{1}{|\mathcal{N}|}\Big(\frac{\partial |\mathcal{N}_2|}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial \mathbf{E}} + \frac{\partial |\mathcal{N}_2|}{\partial \mathbf{E}}\Big) - \frac{|\mathcal{N}_2|}{|\mathcal{N}|^2}\frac{\partial |\mathcal{N}|}{\partial \mathbf{E}},\\ \frac{\partial \xi_2}{\partial \mathbf{E}} &= \frac{1}{|\mathcal{N}|}\Big(\frac{\partial |\mathcal{N}_3|}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial \mathbf{E}} + \frac{\partial |\mathcal{N}_3|}{\partial \mathbf{E}}\Big) - \frac{|\mathcal{N}_3|}{|\mathcal{N}|^2}\frac{\partial |\mathcal{N}|}{\partial \mathbf{E}}.\end{aligned}$$

7.3 The derivatives $\frac{\partial \mathbf{G}}{\partial \mathbf{\Delta}}$

As in Section 7.2, we compute the derivatives of $\mathcal{G}$ with respect to $\mathbf{E}$ and multiply the result by $\frac{\partial \mathbf{E}}{\partial \mathbf{\Delta}}$. We have

$$\frac{\partial \mathcal{K}_j}{\partial \mathbf{e}_j^\rho} = \ddot{\mathbf{q}}_j + \mu \frac{\mathbf{q}_j}{|\mathbf{r}_j|^3}\left(1 - 3\rho_j \frac{(\mathbf{r}_j \cdot \mathbf{e}_j^\rho)}{|\mathbf{r}_j|^2}\right), \qquad j = 1, 2$$

and

$$\frac{\partial \mathcal{K}_j}{\partial \mathbf{e}_j^\alpha} = \frac{\partial \mathcal{K}_j}{\partial \mathbf{e}_j^\delta} = 0, \qquad \frac{\partial \mathcal{K}_1}{\partial \mathbf{E}_2} = \frac{\partial \mathcal{K}_2}{\partial \mathbf{E}_1} = \mathbf{0}.$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{e}_1^\rho}[(\mathbf{L}_1 - \mathbf{L}_2)\cdot \mathbf{v}_2] &= \frac{1}{\mu}\Big[(\mathbf{r}_1 \cdot \mathbf{v}_2)\left(2\dot{\rho}_1 \dot{\mathbf{q}}_1 + \mu\rho_1 \frac{\mathbf{q}_1}{|\mathbf{r}_1|^3}\right) + \\
&\quad + \left(|\dot{\mathbf{r}}_1|^2 - \frac{\mu}{|\mathbf{r}_1|}\right)\rho_1 \mathbf{v}_2 - (\dot{\mathbf{r}}_1 \cdot \mathbf{v}_2)(\dot{\rho}_1 \mathbf{q}_1 + \rho_1 \dot{\mathbf{q}}_1) - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\dot{\rho}_1 \mathbf{v}_2\Big], \\
\frac{\partial}{\partial \mathbf{e}_1^\alpha}[(\mathbf{L}_1 - \mathbf{L}_2)\cdot \mathbf{v}_2] &= \frac{\xi_1}{\mu}[2(\mathbf{r}_1 \cdot \mathbf{v}_2)\dot{\mathbf{q}}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{v}_2)\mathbf{q}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\mathbf{v}_2], \\
\frac{\partial}{\partial \mathbf{e}_1^\delta}[(\mathbf{L}_1 - \mathbf{L}_2)\cdot \mathbf{v}_2] &= \frac{\zeta_1}{\mu}[2(\mathbf{r}_1 \cdot \mathbf{v}_2)\dot{\mathbf{q}}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{v}_2)\mathbf{q}_1 - (\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1)\mathbf{v}_2],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{e}_2^\rho}[(\mathbf{L}_1 - \mathbf{L}_2)\cdot \mathbf{v}_2] &= -\mathbf{L}_1 \times \mathbf{q}_2 + \\
&\quad + \frac{1}{\mu}\Big[(\dot{\rho}_2 \mathbf{q}_2 + \rho_2 \dot{\mathbf{q}}_2)(\dot{\mathbf{r}}_2 \cdot \mathbf{v}_2) + (\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)\mathbf{q}_2 \times \dot{\mathbf{q}}_2\Big], \\
\frac{\partial}{\partial \mathbf{e}_2^\alpha}[(\mathbf{L}_1 - \mathbf{L}_2)\cdot \mathbf{v}_2] &= \frac{1}{\mu}[\xi_2(\dot{\mathbf{r}}_2 \cdot \mathbf{v}_2) + \zeta_2(\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)]\mathbf{q}_2, \\
\frac{\partial}{\partial \mathbf{e}_2^\delta}[(\mathbf{L}_1 - \mathbf{L}_2)\cdot \mathbf{v}_2] &= \frac{1}{\mu}[\zeta_2(\dot{\mathbf{r}}_2 \cdot \mathbf{v}_2) - \xi_2(\dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)]\mathbf{q}_2.
\end{aligned}$$

For Lambert's equation we have

$$\frac{\partial \mathcal{L}}{\partial \mathbf{E}} = \frac{\partial n}{\partial \mathbf{E}}(\bar{t}_1 - \bar{t}_2) + \frac{\partial(\beta - \sin\beta)}{\partial \mathbf{E}} - \frac{\partial(\gamma - \sin\gamma)}{\partial \mathbf{E}},$$

with

$$\frac{\partial n}{\partial \mathbf{E}_1} = -\frac{3}{2\mu}\sqrt{-2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \mathbf{E}_1}, \qquad \frac{\partial n}{\partial \mathbf{E}_2} = \mathbf{0},$$

$$\frac{\partial(\beta - \sin\beta)}{\partial \mathbf{E}} = 2\sqrt{\frac{\Gamma_+}{1-\Gamma_+}}\frac{\partial \Gamma_+}{\partial \mathbf{E}},$$
$$\frac{\partial(\gamma - \sin\gamma)}{\partial \mathbf{E}} = 2\sqrt{\frac{\Gamma_-}{1-\Gamma_-}}\frac{\partial \Gamma_-}{\partial \mathbf{E}}.$$

Moreover

$$\begin{aligned}
\frac{\partial \Gamma_+}{\partial \mathbf{E}_1} &= -\frac{2\mathcal{E}_1}{4\mu}\Big(\frac{\partial r_1}{\partial \mathbf{E}_1} + \frac{\partial d}{\partial \mathbf{E}_1}\Big) + \frac{\Gamma_+}{2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \mathbf{E}_1},\\
\frac{\partial \Gamma_+}{\partial \mathbf{E}_2} &= -\frac{2\mathcal{E}_1}{4\mu}\Big(\frac{\partial r_2}{\partial \mathbf{E}_2} + \frac{\partial d}{\partial \mathbf{E}_2}\Big),\\
\frac{\partial \Gamma_-}{\partial \mathbf{E}_1} &= -\frac{2\mathcal{E}_1}{4\mu}\Big(\frac{\partial r_1}{\partial \mathbf{E}_1} - \frac{\partial d}{\partial \mathbf{E}_1}\Big) + \frac{\Gamma_-}{2\mathcal{E}_1}\frac{\partial(2\mathcal{E}_1)}{\partial \mathbf{E}_1},\\
\frac{\partial \Gamma_-}{\partial \mathbf{E}_2} &= -\frac{2\mathcal{E}_1}{4\mu}\Big(\frac{\partial r_2}{\partial \mathbf{E}_2} - \frac{\partial d}{\partial \mathbf{E}_2}\Big),
\end{aligned}$$

with

$$\frac{\partial(2\mathcal{E}_1)}{\partial \mathbf{E}_1} = \Big(2\dot\rho_1\dot{\mathbf{q}}_1 + 2\mu\rho_1\frac{\mathbf{q}_1}{r_1^3},\ 2\xi_1\dot{\mathbf{q}}_1,\ 2\zeta_1\dot{\mathbf{q}}_1\Big),$$
$$\frac{\partial r_1}{\partial \mathbf{E}_1} = \Big(\frac{\rho_1\mathbf{q}_1}{r_1}, \mathbf{0}, \mathbf{0}\Big), \qquad \frac{\partial r_2}{\partial \mathbf{E}_2} = \Big(\frac{\rho_2\mathbf{q}_2}{r_2}, \mathbf{0}, \mathbf{0}\Big),$$
$$\frac{\partial d}{\partial \mathbf{E}_1} = \frac{\rho_1}{d}(\mathbf{q}_1 - \mathbf{r}_2, \mathbf{0}, \mathbf{0}), \qquad \frac{\partial d}{\partial \mathbf{E}_2} = \frac{\rho_2}{d}(\mathbf{q}_2 - \mathbf{r}_1, \mathbf{0}, \mathbf{0}).$$

8 Alternative known methods

We recall below two already known methods that can be used in place of the algorithm described in Sections 5, 6, 7, with the available data. An important difference is that these methods do not provide corrections to the angles α, δ.

8.1 Gibbs' method

From three position vectors of an observed body at the same pass we can compute an orbit using Gibbs' method, see [4, Chap. 8]. We recall below the formulas of this method. Given the position vectors $\mathbf{r}_j, j = 1, 2, 3$ at times t_j, Gibbs' method gives

$$\dot{\mathbf{r}}_2 = -d_1\,\mathbf{r}_1 + d_2\,\mathbf{r}_2 + d_3\,\mathbf{r}_3,$$

where

$$
\begin{aligned}
d_j &= G_j + H_j r_j^{-3}, \quad j = 1,2,3, \\
G_1 &= \frac{t_{32}^2}{t_{21}\, t_{32}\, t_{31}}\ ,\ G_3 = \frac{t_{21}^2}{t_{21}\, t_{32}\, t_{31}}\ ,\ G_2 = G_1 - G_3, \\
H_1 &= \mu\, t_{32}/12\ ,\ H_3 = \mu t_{21}/12\ ,\ H_2 = H_1 - H_3.
\end{aligned}
$$

Here $t_{ij} = t_i - t_j$, $r_j = |\mathbf{r}_j|$.

8.2 Keplerian integrals

From two radar tracks, we can obtain by interpolation the values of $(\bar{\alpha}, \bar{\delta}, \rho, \dot{\rho})$ at epochs $\bar{t}_1$, $\bar{t}_2$. If we wish to determine the values of the unknowns $\dot{\alpha}, \dot{\delta}$, or equivalently of ξ, ζ defined by (4), we can use the Keplerian integrals method, see [9], [2], [3]. This method uses the equations

$$
\mathbf{c}_1 = \mathbf{c}_2, \qquad \mathcal{E}_1 = \mathcal{E}_2,
$$

which can be explicitly solved, giving at most two solutions.

9 Numerical tests

We have performed some numerical tests with simulated objects, without the J_2 effect, but adding errors to the observations. Here we describe the results of our tests only for one simulated object. We add to the data of the tracks a Gaussian error, with zero mean and standard deviations listed in Table 1. In particular we consider the case where we add no error to the range ρ (RMS (1) in the table). The

RMS	α	δ	ρ
(1)	0.2	0.2	-
(2)	0.1	0.1	5×10^{-3}
(3)	0.2	0.2	10^{-2}

Table 1: RMS of the errors added to the radar tracks.

data that we obtain by interpolation of two radar tracks are displayed in Table 2 for the simulated object. The case labeled with RMS (1) is peculiar, in fact we

Epoch	Data	RMS (1)	RMS (2)	RMS (3)
	$\bar{\alpha}$ (deg)	51.17	51.20	51.18
	$\bar{\delta}$ (deg)	-5.47	-5.44	-5.47
54127.155035	ρ (km)	1984.4	1984.4	1984.4
(MJD)	$\dot{\rho}$ (km/d)	-73313	-73268	-73223
	$\ddot{\rho}$ (km/d^2)	116444362	116534247	116676527
	$\bar{\alpha}$ (deg)	264.30	264.28	264.30
	$\bar{\delta}$ (deg)	-66.77	-66.79	-66.77
54127.582118	ρ (km)	1893.5	1893.5	1893.5
(MJD)	$\dot{\rho}$ (km/d)	-323582	-323712	-323842
	$\ddot{\rho}$ (km/d^2)	123666885	123168829	122613669

Table 2: Data interpolated from the radar tracks of the test object.

interpolate the available values of α, δ and we use the exact values of $\rho, \dot{\rho}, \ddot{\rho}$, that we can compute from the given orbit.
In Table 3 we show the orbits computed by the methods of Gibbs (G), by the Keplerian integrals (KI) and by the infinitesimal angles (InfAng), using the quadratic equations introduced in Section 6.2. For KI and InfAng we use the three data sets displayed in Table 2. Note that InfAng with RMS (1) is able to correct the errors in α, δ and to recover the orbital elements of the known orbit. For RMS (2) and RMS (3), InfAng obtains a better value of the semimajor axis a, and slightly worse values of the other orbital elements, if compared with KI. To be consistent, for KI with RMS (1) we use the exact values of $\rho, \dot{\rho}$. The results with Gibbs' method are not very good. On the other hand it uses only part of the information contained in the data: here we use the three vectors $(t_j, \rho_j, \alpha_j, \delta_j)$ of the first track at epochs t_j, with $j = 1, 2, 4$.
The infinitesimal angles method with linear equations shows some limitations, in a way that we could not compute an orbit for reliable values of the observational errors.

10 Conclusions

We have introduced a new method to compute preliminary orbits of space debris using radar observations. The comparison with already existing methods was performed for some test cases. Large scale tests should be done, to check the performance of the algorithm, possibly with real data. We plan to investigate the case

	Known orbit	OD methods			RMS
		G	KI	InfAng	
		8267.75	7816.61	7818.10	(1)
a	7818.10	8022.40	7815.71	7818.02	(2)
		8267.91	7814.80	7818.09	(3)
		0.005	0.066	0.066	(1)
e	0.066	0.037	0.066	0.065	(2)
		0.005	0.066	0.065	(3)
		70.17	65.85	65.81	(1)
I	65.81	68.02	65.84	65.61	(2)
		70.17	65.85	65.34	(3)
		217.55	216.25	216.25	(1)
Ω	216.25	216.92	216.25	216.30	(2)
		217.55	216.25	216.37	(3)
		336.76	357.00	357.16	(1)
ω	357.16	356.16	357.17	357.43	(2)
		335.93	357.19	357.53	(3)
		219.54	202.29	202.08	(1)
ℓ	202.08	201.46	202.09	201.72	(2)
		220.38	202.09	201.53	(3)

Table 3: Orbital elements at epoch $\bar{t}_1 = 54127.15$ MJD computed by the tracks producing the three data sets of Table 2. Distances are expressed in km, angles in degrees.

which includes the J_2 effect in the equations: this is essential to link radar tracks of LEO orbits after several revolutions.

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12 Appendix

12.1 Lambert's equations for elliptic motion

In this section, following [10] and [7], we summarize the steps to derive Lambert's equation for elliptic motion under a Newtonian force and we give a geometric interpretation of the result. Indeed we obtain four distinct equations per number of revolutions of the observed body. Note that, dealing with radar observations of space debris, the time between two distinct arcs of observations usually covers several revolutions.

Theorem 1 *(Lambert, 1761) In the elliptic motion under the Newtonian gravitational attraction, the time $\Delta t = t_2 - t_1$ spent to describe any arc (without multiple revolutions) from the initial position P_1 to the final position P_2 depends only on the semimajor axis a, on the sum $r = r_1 + r_2$ of the two distances $r_1 = |P_1 - F|$, $r_2 = |P_2 - F|$ from the center of force F, and the length d of the chord joining P_1 and P_2. More precisely we have*

$$n\Delta t = \beta - \gamma - (\sin\beta - \sin\gamma),$$

where $n = n(a)$ is the mean motion, and the angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r+d}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r-d}{4a},$$

and

$$0 \le \beta - \gamma \le 2\pi. \tag{16}$$

Proof. We can assume, without loss of generality, that the positions of the points P_1, P_2 are defined by two values E_1, E_2 of the eccentric anomalies such that $0 \le E_2 - E_1 \le 2\pi$.
The difference of Kepler's equations at the two epochs gives

$$n\Delta t = E_2 - E_1 - e(\sin E_2 - \sin E_1),$$

where e is the orbital eccentricity. From elementary geometrical relations we obtain

$$\begin{aligned}\frac{r}{a} &= 2\left(1 - e\cos\frac{E_1+E_2}{2}\cos\frac{E_2-E_1}{2}\right),\\ \frac{d}{a} &= 2\sin\frac{E_2-E_1}{2}\sqrt{1 - e^2\cos^2\frac{E_1+E_2}{2}}.\end{aligned}$$

It follows that

$$\frac{r+d}{2a} = 1 - \cos\Big(\frac{E_2 - E_1}{2} + \arccos\Big(e\cos\frac{E_2 + E_1}{2}\Big)\Big), \tag{17}$$

$$\frac{r-d}{2a} = 1 - \cos\Big(-\frac{E_2 - E_1}{2} + \arccos\Big(e\cos\frac{E_2 + E_1}{2}\Big)\Big). \tag{18}$$

In particular, for a real elliptical orbit to be possible the given scalar quantities must satisfy the relations $r \geq d$ and $4a - r \geq d$.
If we define

$$\beta_0 = 2\arcsin\left(\sqrt{\frac{r+d}{4a}}\right), \qquad \gamma_0 = 2\arcsin\left(\sqrt{\frac{r-d}{4a}}\right),$$

then, using relation

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}, \qquad \theta \in \mathbb{R},$$

and setting

$$\beta = \frac{E_2 - E_1}{2} + \arccos\Big(e\cos\frac{E_2 + E_1}{2}\Big), \tag{19}$$

$$\gamma = -\frac{E_2 - E_1}{2} + \arccos\Big(e\cos\frac{E_2 + E_1}{2}\Big), \tag{20}$$

we find that the pairs

$$(\beta, \gamma) = (\beta_0, \gamma_0),\ (\beta_0, -\gamma_0),\ (2\pi - \beta_0, -\gamma_0),\ (2\pi - \beta_0, \gamma_0) \tag{21}$$

satisfy equations (17), (18).
Up to addition of the same integer multiple of 2π to both β and γ, the pairs (21) are the only ones fulfilling (17), (18) and (16).
From (19), (20) we obtain

$$\beta - \gamma = E_2 - E_1, \quad \cos\frac{\beta + \gamma}{2} = e\cos\frac{E_2 + E_1}{2},$$

that yields

$$n\Delta t = \beta - \gamma - (\sin\beta - \sin\gamma)\,.$$

In fact

$$\begin{aligned} \sin\beta - \sin\gamma &= 2\sin\frac{\beta - \gamma}{2}\cos\frac{\beta + \gamma}{2} \\ &= 2e\sin\frac{E_2 - E_1}{2}\cos\frac{E_2 + E_1}{2} = e(\sin E_2 - \sin E_1). \end{aligned}$$

□

The pairs (β, γ) given in (21) correspond to 4 geometrically distinct possible paths from the initial to the final position, see Figure 1. Given the points P_1, P_2 and the attracting focus F, for a fixed value a of the semimajor axis, we find two different ellipses passing through P_1 and P_2. They share the attracting focus F, but not the second focus (F_* and F_{**} in the figure). For each ellipse we have two possible arcs from P_1 to P_2, with different orientation, clockwise and counter-clockwise.
The 4 cases are discussed in [6], [7], and are distinguished on the basis of the abscissa of the intercept Q of the straight line through P_1 and P_2, on the axis passing through the foci of one of the ellipses, measured from its center.

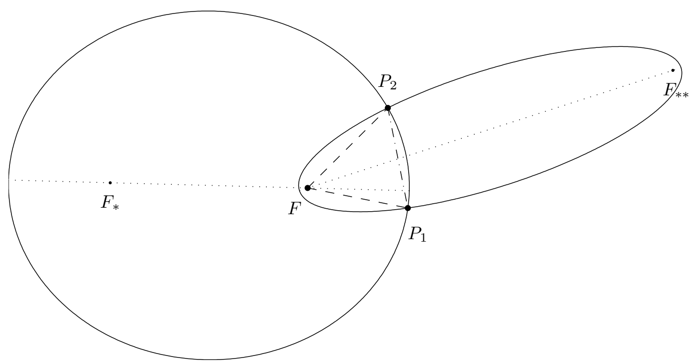

Figure 1: The four cases occurring in Lambert's theorem.

In [7] the 4 cases are also distinguished using the region $\mathcal{R}$ whose border is formed by the arc and the chord joining P_1 and P_2. We use this criterion for the classification given below.
For a complete list of the equations coming from Lambert's theorem, that need to be considered in our problem, we have to take into account the possible occurence of multiple revolutions along the orbit. Denoting by n the mean motion, the following expressions for Δt are obtained:[1]

i) $\Delta t = T_1 + 2k\pi/n$, when the arc covers k revolutions and $\mathcal{R}$ contains neither of the foci;

ii) $\Delta t = T_2 + 2k\pi/n$, when the arc covers k revolutions, $\mathcal{R}$ contains the attracting focus F but not the other one;

[1]Here the region $\mathcal{R}$ is defined ignoring multiple revolutions.

iii) $\Delta t = -T_1 + 2(k+1)\pi/n$, when the arc covers k revolutions and $\mathcal{R}$ contains both foci;

iv) $\Delta t = -T_2 + 2(k+1)\pi/n$, when the arc covers k revolutions, $\mathcal{R}$ does not contain the attracting focus F but contains the other one;

where T_1, T_2 are given by

$$nT_1 = \beta_0 - \gamma_0 - (\sin\beta_0 - \sin\gamma_0),$$
$$nT_2 = \beta_0 + \gamma_0 - (\sin\beta_0 + \sin\gamma_0).$$

The four cases above can be summarized in the equation

$$n\Delta t = \beta - \gamma - (\sin\beta - \sin\gamma) + 2k\pi, \quad k \in \mathbb{N},$$

where the angles β, γ are defined by

$$\sin^2\frac{\beta}{2} = \frac{r+d}{4a}, \qquad \sin^2\frac{\gamma}{2} = \frac{r-d}{4a},$$

and

$$0 \le \beta - \gamma \le 2\pi.$$

We also observe that in [8] there is a geometrical interpretation for the angles β, γ.

12.2 Corrections to the observations

We describe a procedure that could be used to correct the angular positions of a track by a pure geometrical argument.
Assume we have the geocentric position vectors $\mathbf{r}_j = \rho_j\mathbf{e}^\rho_j + \mathbf{q}_j$, $j = 1\ldots 4$, with $\mathbf{q}_j$ the geocentric positions of the observer, from the radar observations of the celestial body. The vectors $\mathbf{r}_j$ would be coplanar, if the orbit were perfectly Keplerian. In general this holds only approximately, due to the observational errors and to the perturbations which should be added to Kepler's motion. We wish to correct these position vectors and define coplanar vectors $\mathbf{r}'_j$, which are slightly different from $\mathbf{r}_j$ and keep the measured value ρ_j of the topocentric radial distances.

In the attempt to define a good approximation of the plane of this idealized Kepler motion, we compute the minimum of the function

$$\boldsymbol{\nu} \mapsto Q(\boldsymbol{\nu}) = \sum_{j=1}^{4}(\mathbf{r}_j \cdot \boldsymbol{\nu})^2$$

with the constraint $|\boldsymbol{\nu}| = 1$. We obtain the equation

$$\sum_{j=1}^{4}(\mathbf{r}_j \cdot \boldsymbol{\nu})\mathbf{r}_j - \lambda\boldsymbol{\nu} = \mathbf{0}, \tag{22}$$

with the Lagrange multiplier $\lambda \in \mathbb{R}$, and consider the solution $\boldsymbol{\nu}_{min}$ of (22) relative to the minimum eigenvalue λ_{min}. We take $\mathbf{e}^{\nu} = \boldsymbol{\nu}_{min}$ as the direction of the Kepler motion plane, denoted by Π_{ν}.
Then, for each $j = 1 \ldots 4$, we rotate the vectors $\boldsymbol{\rho}_j = \rho_j \mathbf{e}^{\rho}_j$ into a vector $\mathcal{R}\boldsymbol{\rho}_j$ as follows (see Figure 2).

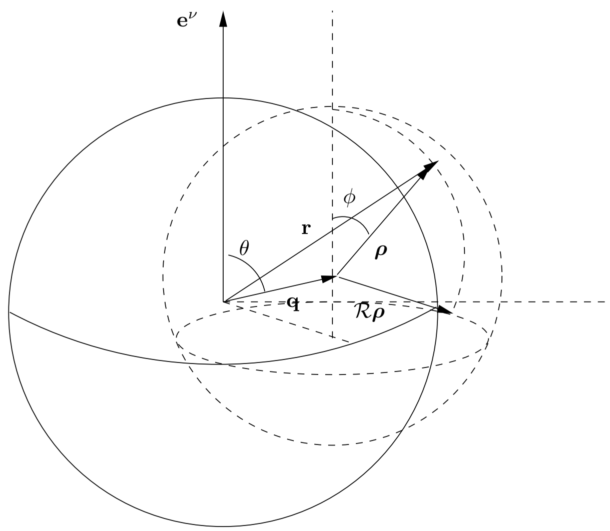

Figure 2: Sketch of the correction of the line of sight. Here we skip the index j.

Since we want to minimize the change in the line of sight, i.e. the observation direction $\mathbf{e}^{\rho}_j$, we rotate the latter around the axis orthogonal to the plane generated by $\mathbf{e}^{\nu}, \mathbf{e}^{\rho}_j$ to reach the plane Π_{ν}. In this way we draw a geodesic arc on the sphere with radius ρ_j, centered at the observer position defined by $\mathbf{q}_j$. This arc joins the position of the observed body with the plane Π_{ν}.

To describe this procedure in coordinates we introduce the angles $\theta_j, \phi_j \in [0, \pi]$ defined by

$$\cos\theta_j = \frac{\mathbf{e}^{\nu} \cdot \mathbf{q}_j}{q_j}, \qquad \cos\phi_j = \mathbf{e}^{\nu} \cdot \mathbf{e}^{\rho}_j.$$

The rotated vector $\mathcal{R}\boldsymbol{\rho}_j$ can be expressed as the linear combination

$$\mathcal{R}\boldsymbol{\rho}_j = A_j \mathbf{e}^{\nu} + B_j \mathbf{e}^{\rho}_j,$$

with $B_j \geq 0$ (since we do not want to rotate the line of sight by more than 90 degrees).
Now set the following conditions:

$$\text{(i)} \quad |\mathcal{R}\boldsymbol{\rho}_j| = \rho_j,$$

$$\text{(ii)} \quad [\mathbf{q}_j + \mathcal{R}\boldsymbol{\rho}_j] \cdot \mathbf{e}^\nu = 0.$$

We obtain

$$A_j^2 + B_j^2 + 2A_jB_j\cos\phi_j = \rho_j^2,$$
$$q_j\cos\theta_j + A_j + B_j\cos\phi_j = 0.$$

From the second equation we obtain

$$A_j = -q_j\cos\theta_j - B_j\cos\phi_j,$$

that substituted into the first yields

$$B_j = \frac{1}{\sin\phi_j}\sqrt{\rho_j^2 - q_j^2\cos^2\theta_j},$$

so that

$$A_j = -\left(q_j\cos\theta_j + \cot\phi_j\sqrt{\rho_j^2 - q_j^2\cos^2\theta_j}\right).$$

We observe that this procedure works provided

$$\rho_j \geq q_j\cos\theta_j, \qquad \text{for } j = 1\ldots4.$$

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